# Comparison of computations of asymptotic flow models in a constricted channel.

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**Abstract**

We aim at comparing computations with asymptotic models issued from incompressible Navier-Stokes at high Reynolds number: the Reduced Navier-Stokes/Prandtl (RNS/P) equations and the Double Deck (DD) equations. We treat the case of the steady two dimensional flow in a constricted pipe. In particular, finite differences and finite element solvers are compared for the RNS/P equations. It results from this study that the two codes compare well. Numerical examples also illustrate the interest of these asymptotic models as well as the flexibility of the finite element solver.

## 1 Introduction

The 2D flow between parallel plates (or as well in a symmetrical pipe) with a constriction mimics a lot of flows of interest, among them one may think of blood flow in a stenosis [3, 19, 20], respiratory flow [21, 27, 24], pipes in the gas or oil industry [18]. Those kind of simplifications arising in the Saint-Venant equations as well (even if here we do not deal with moving interfaces). Of course, this problem may be considered as very classical with no more numerical difficulties. It may be solved by various numerical methods for the Navier Stokes equations [10, 32, 11, 34].

Therefore, we will focus on model equations issued from Navier-Stokes, when a long wave approximation may be done. Those equations are obtained by change of scales, so that vanishing small terms are removed from the complete set of Navier-Stokes equations.

This approach is known since Prandtl pioneering work in 1905 (Prandtl [22], Schlichting [25]) and several books now deal with asymptotic flow models (Schlichting & Gersten [26], Cousteix & Mauss [6], Sobey [29], Sychev et al. [30]).

The obtained equations are the Prandtl equations with different boundary conditions. So, one of the major difficulties is that those equations are not standard compared to Navier Stokes or Euler equations, for which a huge variety of numerical methods have been developed. As a result, most of the numerical methods designed for Navier-Stokes fail to solve these equations, which in general present a parabolic character, and well-suited methods are not so much developed [13, 29, 14].

So, we will present in the first part §2 some asymptotic models issued from the Navier Stokes equations. Those are the so called Reduced Navier-Stokes/Prandtl (RNS/P) (Lagrée & Lorthois [15]) and the Double Deck (DD [28]) equations. In the second part §3, we will describe three numerical techniques that have been developed for these equations and that will be compared. The first is a finite difference method inspired from the classical procedures to solve parabolic equations, such as the heat equation [9, Ch. 6]. It has been developed and used in various contexts [15, 16, 33, 17, 5]. The second one is the "Keller Box" technique, which is also a finite difference method [13, 4]. The last one is a finite element method, which main features are a low order approximation of the pressure and/or a grad-div stabilization term. It has been developed more recently [2] and for this reason needs to be validated through comparison with existing solutions or reference codes. Therefore, in a third part §4, we present numerical comparisons of those approaches. Furthermore, we confront simulations with the simplified model equations (RNS/P, DD) to simulations with the full Navier Stokes equations. In §5, we finally draw conclusions about the experiments and discuss the interest of both these models and methods. Concerning the skin friction and the pressure distribution at the wall, we will show that the results are similar, validating the models and the methods.

# 2 Mathematical flow models

## 2.1 Geometry and hypotheses

We want to compute the velocity and pressure field in a geometry like those of Figure 1. It consists in two parallel plates with a symmetrical indentation. Note that the axi-symmetrical case is exactly the same in principle. The flow is supposed newtonian, laminar, incompressible and steady. It mainly goes from the inlet to the outlet. The base flow in the case of no indentation is the Poiseuille flow. We use the height $h$ as scale and the velocity scale $U_0$ is such that the non dimensional Poiseuille flow reads

$$u(0, y) = (1 - y)y,\ v = 0.$$

The Reynolds number is defined with $h$, $U_0$ and $\nu$ the constant viscosity: $Re = \frac{U_0 h}{\nu}$. It is supposed very large, but the flow is supposed to remain laminar.

The asymptotic models issued from high Reynolds number Navier-Stokes (NS) are the Reduced Navier-Stokes/Prandtl (RNS/P) equations and the Double Deck (DD) equations.

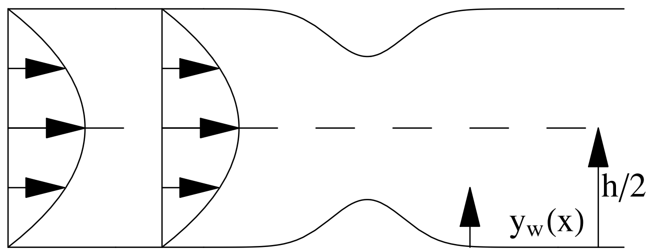


Figure 1: The incompressible 2D flow between two plates, the lower plate is in $y = 0$, the upper in $y = h$. A symmetrical indentation is given at the lower wall as $y = y_w(x)$ and at the upper wall as $y = h - y_w(x)$. As the problem is symmetrical it is solved between $y_w(x)$ and $h/2$.

### 2.2 Navier-Stokes equations

The problem is to solve the Navier-Stokes equations written without dimensions:

$$\begin{cases} u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{\partial p}{\partial x} + \dfrac{1}{Re}(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}), \\ u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = -\dfrac{\partial p}{\partial y} + \dfrac{1}{Re}(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}), \\ \quad \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0. \end{cases} \tag{1}$$

The boundary conditions are no-slip condition on the walls, a condition of symmetry, the initial velocity profile and the value of the pressure at the outlet:

$$u = v = 0 \;\text{ on } y = y_w(x),\; \frac{\partial u}{\partial y} = 0 \;\text{ on } y = 1/2,\; u(0, y) = (1 - y)y \text{ and } p(x_{out}, y) = 0. \tag{2}$$

### 2.3 RNS/P

Looking at long bumps of scale $L_b$, with $L_b >> h$, we may expand Navier-Stokes using this longitudinal scale and keeping transversally the $h$ scale. So, using $x^* = L_b x$, $y^* = hy$, $u^* = U_0 u$, $v^* = V_0 v$ and $p^* = p_0 + \rho U_0^2 p$, we obtain a specific case when $L_b = hRe$ and $V_0 = U_0/Re$. With those scales, we obtain in fact the Prandtl equations. It means that the second order derivative term in $x$ in the equations disappears because it is of order $Re^{-2}$ and the pressure remains constant across the section because the transverse pressure gradient is of order $Re^{-1}$. We now formally write back in the scales $x^* = hx$, $y^* = hy$, $u^* = U_0 u$, $v^* = U_0 v$ this system and we obtain:

$$\begin{cases} u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{\partial p}{\partial x} \quad + \dfrac{1}{Re}\dfrac{\partial^2 u}{\partial y^2}, \\ 0 = -\dfrac{\partial p}{\partial y}, \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0. \end{cases} \tag{3}$$

The boundary condition are the no-slip condition on the wall, a condition of symmetry, the initial velocity profile and the value of the pressure at the inlet:

$$u = v = 0 \text{ on } y = y_w(x),\ \frac{\partial u}{\partial y} = 0 \text{ on } y = 1/2,\ u(0,y) = (1-y)y \text{ and a given } p(x_{in}, y) \tag{4}$$

Note first that the total pressure drop $p(x_{in}, y) - p(x_{out}, y)$ is a result of the computation, so thereafter we use the output pressure as reference of pressure $p(x_{out}, y) = 0$. This is in general not the case in standard boundary layer theory [26] where the pressure is a given function. Note also that if we add the missing second order derivative $\frac{1}{Re}\frac{\partial^2 u}{\partial x^2}$ in $(3)_1$, the equations we obtain are the so called primitive equations used in oceanography [1].

## 2.4 Double Deck

### 2.4.1 Equations

Looking at small bumps of scale $\varepsilon$, with $\varepsilon \ll 1$, we may expand Navier-Stokes using this transversal scale and using longitudinally any $L$ scale (consistant with the bump length). So, using $x^* = Lx$, $y^* = \varepsilon hy$, $u^* = \varepsilon U_0 u$, $v^* = (\varepsilon^2 h/L)U_0 v$ and $p^* = p_0 + \rho\varepsilon^2 U_0^2 p$, we obtain a specific case when $\varepsilon = (L/h)^{1/3} Re^{-1/3}$. With those scales, we obtain in fact the Prandtl equations, but they are called the Double Deck equations [28]. In the Lower Deck:

$$\begin{cases} u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{\partial p}{\partial x} \quad + \dfrac{\partial^2 u}{\partial y^2}, \\ 0 = -\dfrac{\partial p}{\partial y}, \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0. \end{cases} \tag{5}$$

The boundary conditions are no-slip condition at the wall, the initial linear velocity profile far upstream and the matching condition at infinity:

$$u = v = 0 \text{ on } y = y_w,\ u \to y \text{ when } x \to -\infty, \text{ and } u \to y \text{ when } y \to \infty. \tag{6}$$

Exactly the same set of equations may be written near the upper wall. The axis $y$ being upside down. In the core flow (the Main- Deck) there is no perturbation.

### 2.4.2 Remark: the linear solution

We note that in this case we have a simple analytical solution in Fourier space obtained by linearisation (Smith [28]), with $TF$ and $TF^{-1}$ the direct and reverse Fourier transforms:

$$\tau = U_0' + U_0'(3Ai(0))(U_0')^{1/3}TF^{-1}[(-ik)^{1/3}TF[y_w]] \tag{7}$$

$$p = (U_0')^2(3Ai'(0))(U_0')^{-1/3}TF^{-1}[(-ik)^{-1/3}TF[y_w]]. \tag{8}$$

where $Ai(x)$ is the Airy function, $Ai(0) = 0.355028$ and $Ai'(0) = -0.258819$, with $U_0' = 1$ the value of the slope at the wall.

### 2.4.3 Remark: transverse pressure gradient

Just to explain some features of the non symmetrical case, we recall here that in the non symmetrical case there is a transverse pressure gradient in the Main- Deck. This is the case for example when no bump is present on the upper wall. This transverse pressure gradient is due to the displacement of the stream lines and we obtain that the length of the bump must be $hRe^{1/7}$ and then $\varepsilon = Re^{-2/7}$. Hence in the Main-Deck, we introduce the classical displacement fonction $-A$ of the streamlines (Smith [28]), then the transverse momentum equation issued from (1):

$$U_0(y)\frac{\partial v_1}{\partial x} = -\frac{\partial p_1}{\partial y}. \tag{9}$$

so that as $v_1 = A'(x)U_0(y)$ and $p = Re^{-2/7}p_1$, the velocity is $u = U_0(y) + Re^{-2/7}A(x)U_0'(y)$ and $v = -Re^{-3/7}A'(x)U_0(y)$, so across the channel there is a pressure drop:

$$p_1(x,1) - p_1(x,0) = -A''(x)\int_0^1 U_0^2(y)dy, \tag{10}$$

and two systems of equations like (5) interact: a first one near the lower wall, but the matching condition at infinity is changed as : $u \to y + A$ when $y \to \infty$ and the pressure is $p = p_1(x,0)$ in the layer near the lower wall. The upper layer near the upper wall (with $y^* = h - \varepsilon hy$), has again equations like (5), but the matching condition at infinity is changed as : $u \to y - A$ when $y \to \infty$. The pressure is $p = p_1(x,1)$ in this layer. The symmetrical case corresponds to $A = 0$ which means that there is no displacement of the stream lines in the Main Deck.

# 3 Numerical methods

The presented methods aim at solving the RNS/P equations. Note however that they are also appropriate for the DD equations, through a straightforward change of boundary conditions.

## 3.1 Finite Differences

The baseline of this Finite Differences method is to solve the problem (3) as an heat equation $u\frac{\partial u}{\partial x} + ... = \frac{\partial^2 u}{\partial y^2} + ...$ As the problem is a kind of heat equation, a marching procedure in $x$ seems natural (like time-marching for the heat equation). The system is indeed parabolic, whereas Navier-Stokes equation present generally either an elliptic (at low Reynolds) or hyperbolic (at high Reynolds) character. Knowing the solution $u_{i,j}$ at station $x = i\Delta x$ for all transverse positions $y = j\Delta y$, the solution $u_{i+1,j}$ at next station $x = (i\Delta x) + \Delta x$ is to be found for all $j$. The second order derivative is implicited, a first guess for the pressure is $p^e_{i+1} = p_i$ and the previous step transverse velocity ($v_{i,j}$) is used. We have a tridiagonal system to solve which is done by Thomas algorithm [31]. This gives an estimation of the velocity $u^e_{i+1,j}$. From this estimation the incompressibility is solved by simple Euler integration in $j$ starting from $v^e_{i+1,j=0} = 0$. Then $v^e_{i+1,j}$ is obtained. But, this velocity does not satisfy the no slip velocity in $y = 2$. So we have to iterate on the value of the pressure $p^e_{i+1}$ (by a Newton algorithm) in order to have a zero velocity at the upper side of the grid. The procedure breaks down if there is reverse flow. Nevertheless, we are able to compute reverse flow cases when $u < 0$ by putting $u^{i,j} = 0$, this is known as the FLARE algorithm [23]. This method has been used in [15, 16, 33, 17, 5]. The same technique is used for the RNSP and the DD cases, only the boundary condition is changed.

## 3.2 Keller Box

The problem (3) can be solved with the "Keller Box" technique (see e.g. [4] or [13]). This is also a Finite Differences marching scheme. The equations are written in introducing only first order derivatives:

$$\frac{\partial u}{\partial y} = G, \quad \frac{\partial G}{\partial y} = u\frac{\partial u}{\partial x} + ...$$

then, the derivatives are centered in the "box" of corners $(i-1, j-1)$ $(i-1, j)$ $(i, j-1)$ and $(i, j)$. The values in $(i-1, j-1)$ $(i-1, j)$ are known. For example $\frac{\partial u}{\partial y} = G$, reads $\frac{u(i,j)-u(i,j-1)}{\Delta y} = \frac{G(i,j)+G(i,j-1)}{2}$.

In fact we need four variables, $\psi$ the stream function, $G$ the shear and $W$ a fictionous variable such as $\frac{\partial p}{\partial x} = -\frac{\partial (W^2/2)}{\partial x}$ (denoted as *Mechoul* approach by Cebeci & Keller) so that

Prandtl equations are:

$$\left\{\begin{array}{ll}\dfrac{\partial\psi}{\partial y}=u, & \dfrac{\partial u}{\partial y}=G,\\[2mm] \dfrac{\partial G}{\partial y}=-\dfrac{\partial(W^2/2)}{\partial x}+u\dfrac{\partial u}{\partial x}-G\dfrac{\partial\psi}{\partial x}, & \dfrac{\partial W}{\partial y}=0.\end{array}\right. \tag{11}$$

As among others there are non linear terms, so $u\frac{\partial u}{\partial x}$ is discretized in

$$\frac{\left(\frac{u(i,j)+u(i-1,j)}{2}+\frac{u(i,j-1)+u(i,j-1)}{2}\right)}{2}\frac{\left(\frac{u(i,j)-u(i-1,j)}{\Delta x}+\frac{u(i,j-1)-u(i-1,j-1)}{\Delta x}\right)}{2},$$

and then a Newton iteration is necessary. Writing the new step $n+1$ as a small increase of the preceding: $u^{n+1}(i,j)=u^n(i,j)+\delta u^n(i,j)$, we obtain a block tridiagonal system: $\delta u^n(i,j)$, $\delta G^n(i,j)$ *etc* are solved by Thomas algorithm [31].

Boundary condition at the wall and at the entrance are simple. In the case of Double Deck (DD), at the top of the domain the velocity is equal to $y$ so that the third equation of the sytem (11) becomes $0=-\frac{\partial(W^2/2)}{\partial x}+u\frac{\partial u}{\partial x}-\frac{\partial\psi}{\partial x}$. Its integral $-W^2+u^2-2\psi$ is then a constant at the top of the domain. This last expression is linearised to obtain the relation in $j=J$ the last line

$$J\Delta y\delta u^n(i,J)-\delta\psi^n(i,J)-W^n(i,J)\delta W^n(i,J)=0.$$

A last important trick is to introduce again the so called FLARE (introduced in [23]) approximation: $u\frac{\partial u}{\partial x}$ is put to 0 when $u<0$.

### 3.3 Finite Elements

Finally, the system (3) can also be solved with a Finite Element method, which main features are similar to mixed methods for the incompressible Navier-Stokes equation. The adaptation of such methods is however not straightforward and requires an appropriate choice of finite element spaces / stabilization terms. The method described here has been proposed and analyzed in [2]. We call $\Omega$ the domain in which the equations are solved; $\Omega$ is supposed to be a polygonal domain in $\mathbb{R}^2$, with boundary $\partial\Omega$; $\Gamma_i\subset\partial\Omega$ is the entry (inlet flow), $\Gamma_w\subset\partial\Omega$ is the rigid wall with no-slip boundary conditions (case of the RNS/P equations) and $\Gamma_o\subset\partial\Omega$ is the exit (outlet flow). Let $\boldsymbol{H_h}$ be the finite element discretization space for the velocity and $\Pi_h$ be the finite element space for the pressure. The discrete variational problem reads: *Find* $(u,v,p)\in\boldsymbol{H_h}\times\Pi_h$, $u=u^0$ *on* $\Gamma_i$, $u=0,v=0$ *on* $\Gamma_w$ *such that:*

$$\left\{\begin{array}{l}\displaystyle\int_\Omega\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)\zeta+\frac{1}{Re}\int_\Omega\frac{\partial u}{\partial y}\frac{\partial\zeta}{\partial y}+\int_\Omega\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\left(\frac{\partial\zeta}{\partial x}+\frac{\partial\xi}{\partial y}\right)\\ \displaystyle\qquad\qquad\qquad\qquad-\int_\Omega p\left(\frac{\partial\zeta}{\partial x}+\frac{\partial\xi}{\partial y}\right)+\int_\Omega q\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0,\end{array}\right. \tag{12}$$

*for all* $(\zeta, \xi, q) \in \boldsymbol{H_h}^{\mathbf{0}} \times \Pi_h$.

$\boldsymbol{H_h}^{\mathbf{0}}$ is the subspace of $\boldsymbol{H_h}$ with functions of vanishing trace on $\Gamma_i \cup \Gamma_w$. A continuation strategy which consists in increasing progressively the inlet velocity is used to solve the problem (12). The non-linearity due to the convection term is treated thanks to the Newton method. At each step of the Newton loop, a multi-frontal Gauss LU factorization [8], implemented in the package UMFPACK [7], permits to solve the linearized discrete problem. The numerical method has been implemented in the framework of the open source finite element software FreeFEM++ [12]. Finally, minor changes have been provided here in comparison to the original method described in [2]. Those are the following:

- as it can be seen in (12), both the symmetrical and antisymmetrical parts of the convection term have been kept, and thus convection is discretized in a natural way (the symmetrical part of the convection was removed in [2] for the purpose of the analysis).
- the Taylor-Hood element with a quadratic interpolation of the velocity [9] has been chosen instead of the $\mathbb{P}_2/\mathbb{P}_1/\mathbb{P}_0$ element suggested in [2], so as to allow a better approximation of the pressure and of the shear stress. Let us emphasize that for this element, the grad-div stabilization is strictly necessary so that the problem admits a solution [2, Remark (1) p.60][1]. As a result, we put $\lambda > 0$.
- mesh refinement may be carried out after the first step of the continuation loop, to enhance the precision of the computation near the bump. A variable metric/Delaunay meshing algorithm based on the Hessian of the velocity/pressure, with a constraint of mesh isotropy, has been used for this purpose [12].

# 4 Numerical experiments

In these sections, we compute with the different methods the flow over a simple gaussian $y_w(x) = \alpha e^{-(x-5)^2}$ bump. For the Finite Differences method the numerical parameters are : $\Delta x = .0125$ and $\Delta y = .005$ whereas the KB uses $\Delta x = .025$ and $\Delta y = .06$, the size of the domain is 12 or more.

For the Finite Element method, the mesh is an unstructured mesh generated automatically by FreeFEM++, with mesh refinement near the bump. The number of continuation iterations is 5. At each continuation iteration, Newton loops are carried out until convergence (with a convergence criterion of $10^{-10}$ on the norm of the velocity). The stabilization parameter $\lambda$ is fixed to 1 and its value is motivated by the numerical studies in [2]. See on figure 2 examples of pressure and perturbation of skin friction from the basic state with different mesh sizes, which show that the method converges when the mesh size is reduced.

[1] A straightforward adaptation of Lemmas 1 and 3 in [2] also shows that the discrete problem for RNS/P admits a solution with any kind of inf-sup stable element for Stokes and a grad-div stabilization.

## 4.1 Double Deck examples

On figure 3 we compare the linearised analytical solution of the Double Deck equations (eq. (7) for $\tau$ and (8) for $p$), the Finite Differences, the Keller Box and the Finite Elements numerical solutions for $\alpha = 0.2$. We draw the perturbation of skin friction $\tau - U_0'$ and the perturbation of pressure. The fact that the skin friction is extremal before the crest of the bump and decreases after the crest is a classical observation. A pressure drop is associated, the minimum of pressure is after the crest of the bump. The pressure is nearly the same for the four methods, except the FE which increases a bit the pressure drop. The Keller Box method and the Finite Difference method give superposed results. The FE solution is a bit jagged for the skin friction obtained by derivation of the velocity due to the choice of the elements (continuous interpolation with Lagrange elements). As $\alpha = 0.2$ is not so small, there is a small difference between the three numerical non linear resolutions and the linear resolution by Fourier transform.

On figure 4, we increase the height $\alpha$ of the bump and explore the nonlinearities of the Double Deck problem. We even have separation of the flow ($\tau$ becomes negative after the bump crest). We compare Finite Differences, Keller Box and Finite Elements. The two first are again very similar, though not exactly the same. The pressure is a bit different for the FE computation. The differences in this part are maybe due to the FLARE approximation (put $u = 0$ when $u < 0$) as the implementation is slightly different in KB and FD, and as there is no such approximation in FE.

On figure 5, we increase much more the height $\alpha$ of the bump and explore the nonlinearities of the Double Deck problem for large bumps. Up to now, the Finite Differences resolutions were done with the FLARE simplification, which is unnecessary with the new Finite Element method (see [2]). So the reverse flow is more accurately computed. We observed previously that the skin friction is extremal before the crest, and decreases after. It may then be negative (for $\alpha \gtrsim 2.22$). When it is negative, it means that we have separation and there is a bulb of recirculation after the bump. For moderate values of $\alpha$ (in fact $\alpha \lesssim 3$), the skin friction becomes negative, reaches a minimum and it increases gently after. It becomes positive again, and then asymptotically goes to the undisturbed initial value. When $\alpha$ is more increased, after $\alpha \simeq 3$, the behavior of the skin friction changes. The skin friction becomes negative but it presents a flat part. Then, for large $\alpha$ a kink appears; after this minimum, the skin friction increases again. But it does not become positive, it reaches an extremum. Then it decreases again, it reaches a minimum and increases again gently, passes 0 and goes asymptotically to 1. The advantage of the finite element method is then clear as it allows large bump size. It is then better than the FD and KB which allow to compute a smaller separation bulb (associated to a smaller bump).

### 4.2 RNSP example

Those preceding comparisons at Double Deck scales enabled us to validate the numerical method. We now compare the different asymptotic models and Navier-Stokes itself. So, on figure 6 we draw the skin friction for a value of $\alpha = 0.1$, and on figure 7 we draw the pressure (the zero value of the pressure has been put at the origin). We draw this for the asymptotic models (Double Deck Keller Box and RNS/P Finite Differences and Finite Elements) and for Navier Stokes (Finite Elements) as well.

Note that the value of the height of the bump in the DD scale must be multiplied by $Re^{1/3}$ due to the change of scale. Note as well that the pressure of Double Deck is multliplied by $Re^{-2/3}$ and that the basic Poiseuille pressure has been added to the Double Deck pressure solution so that $-2x/Re + Re^{-2/3}p$ is plotted.

The results of all the models are very similar, even for this moderate value of the Reynolds number ($Re = 750$). We point out here that we choose the definition of the Reynolds number to have a unit slope at the wall. But constructed with the maximum value of the velocity the Reynolds is $R_m = Re/4$ and constructed with the average value of the velocity the Reynolds is only $R_a = Re/6$. So even for a rather small Reynolds the result of the asymptotic methods are very similar to Navier Stokes.

## 5 Discussion and perspectives

We presented and cross-compared different numerical methods to compute the laminar steady viscous flow between two plates with symmetrical indentation. Of course, the Navier Stokes numerical effort is nowadays not very strong, but here, we presented equations which break the elliptic character of Navier Stokes, those equations are parabolic.

We have an asymptotic hierarchy: first at infinite Reynolds number, we have the Double Deck description; in fact this description is included (see [16]) in the Reduced Navier Stokes asymptotic description, which is itself included in the Navier Stokes equations. The interest of asymptotical models is that they focus on the more important terms in the equations: they show that the most important terms are the longitudinal convection and the transverse diffusion, and also that the pressure can be considered approximatively constant in every transverse section.

We validated the different algorithms: two Finite Difference methods [15, 13] and a Finite Element method issued from [2]. The interest of the Finite Element method is clearly that it allows to get rid of the FLARE approximation used in the Finite Differences marching methods. So, we presented original examples of massive separation after of large bump. A drawback of the Finite Element method is that it is not so much faster than the complete Finite Element Navier Stokes resolution (see [2]), and this point is object of current research work. At the opposite, the Finite Differences marching methods are very fast as they are marching in $x$, and they are enough precise to tackle with moderate separation bulbs. The comparison with the FE method shows that the FLARE approximation is in fact enough precise, validating the approximation.

Interestingly enough, our implementation allows us to play with the terms, for example we can reintroduce part of the transverse equation from (1), for instance we reintroduce only:

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} \tag{13}$$

which is nearly (9) at large Reynolds. Then as expected from linear theory of flow in non symmetrical channels [29], there is an upstream effect. This means that upstream the bump, the skin friction and the velocity are disturbed. We can see this on figure 8 where RNSP with the transverse equation (13) are solved and compared to Navier Stokes. This opportunity to remove terms in Navier Stokes is very promising and opens the possibility to a lot of comparisons of asymptotic models at large $Re$ versus full Navier Stokes.

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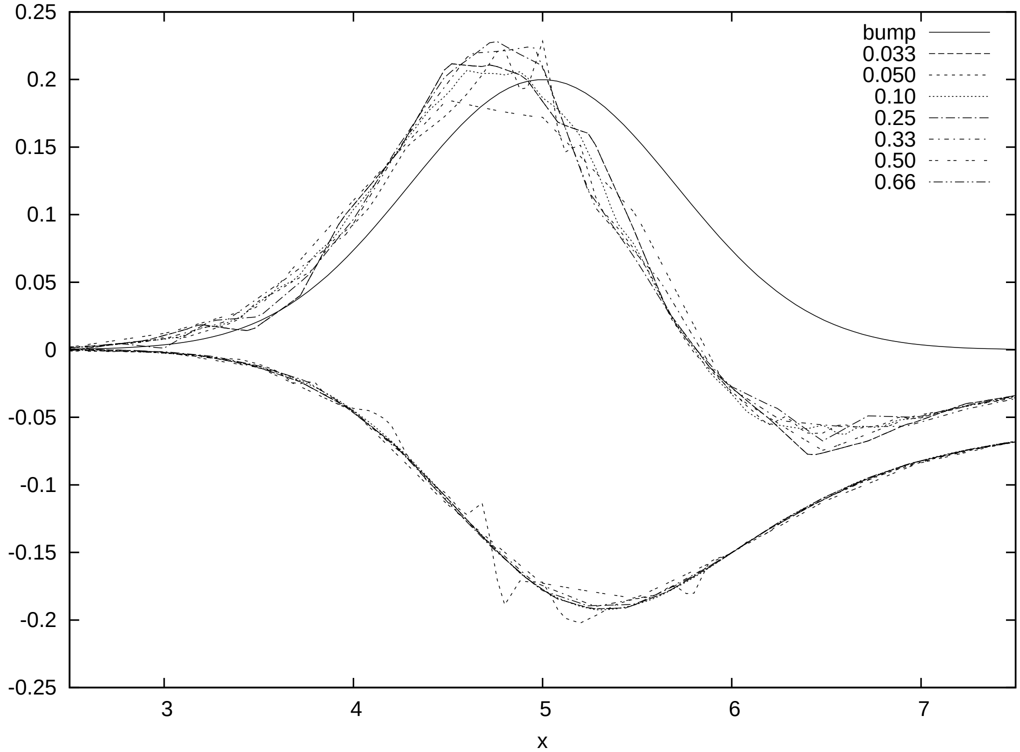


Figure 2: The gaussian bump of height $\alpha = 0.2$, the perturbation (to emphasize the effect) of the skin friction ($\tau - 1$) and the pressure, in the Finite Element case (FE). Influence of the size of the smaller elements (0.66, 0.5, 0.25, 0.1 0.05 and 0.033).

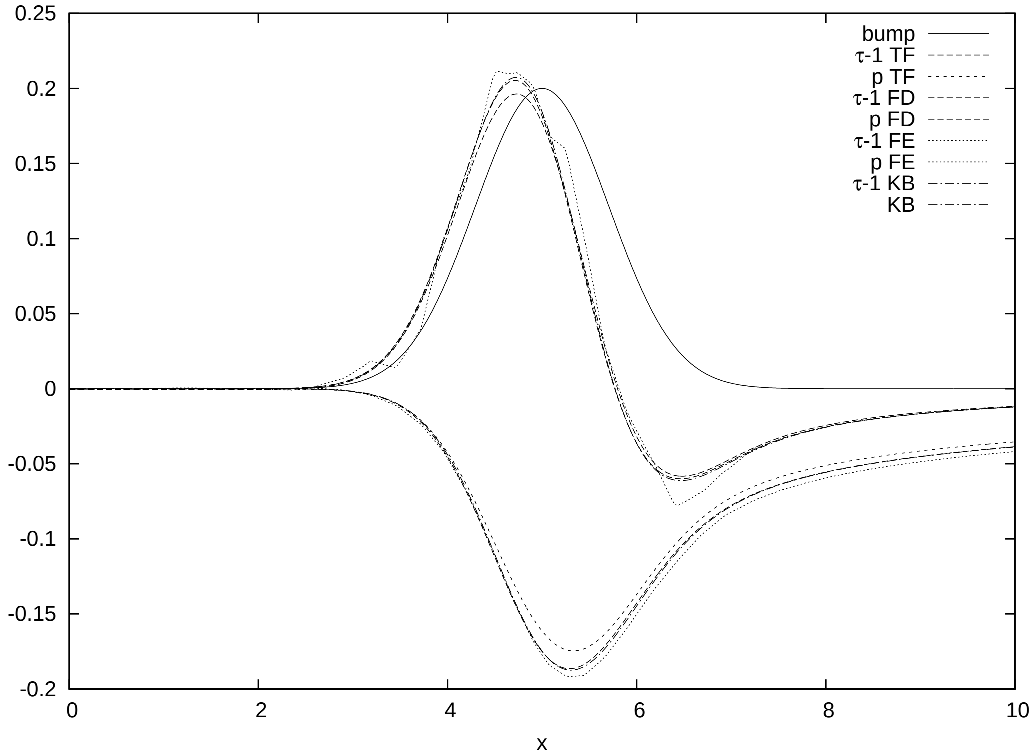


Figure 3: The bump, the perturbation (to emphasize the effect) of the skin friction $(\tau - 1)$ and the pressure, in the Double Deck case. The height of the bump is 0.2. Comparison of the (FD KB and FE), and linear analytical expression DD eq. (7) for $\tau$ and (8) for $p$. The pressure and skin friction are nearly the same for the four methods. The results of KB and FD are superposed. The FE code increases a little the pressure drop. For the skin friction, the FE solution is a bit jagged. The linear solution differs slightly but it is due to the fact that 0.2 is not so small.

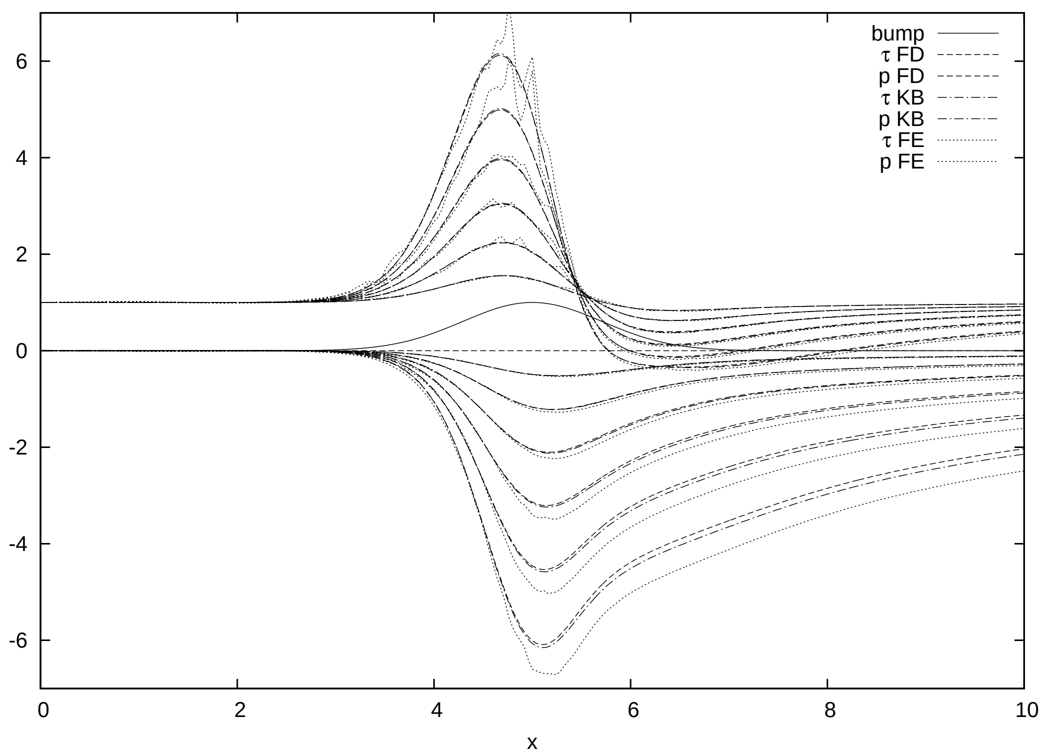


Figure 4: The bump, the skin friction ($\tau$) and the pressure as the height of the bump increases (size of the bump from 0.5, 1.0, 1.5, and 2.0) in the Double Deck case. Comparison of FD, KB and FE. There is a very small difference between FD and KB, and a noticeable between those two and FE. It is likely that we observe the effect of the FLARE approximation.

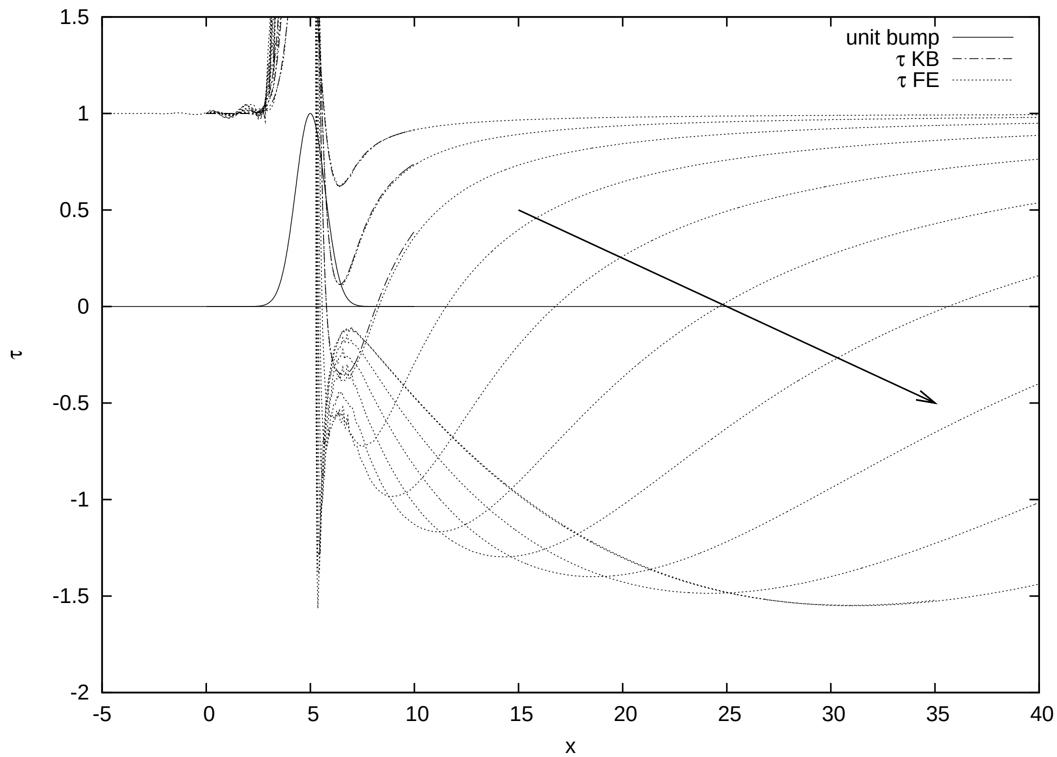


Figure 5: Reduced skin friction $\tau$ in the Double Deck case DD for large height of the bump. FEM: $\alpha = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ and KB $\alpha = 1, 2, 3$. Arrow in the direction of increasing $\alpha$. KB fails for $\alpha > 3.5$, for this value, a sharp kink begins to appear and a maximum of the negative skin friction appears as well.

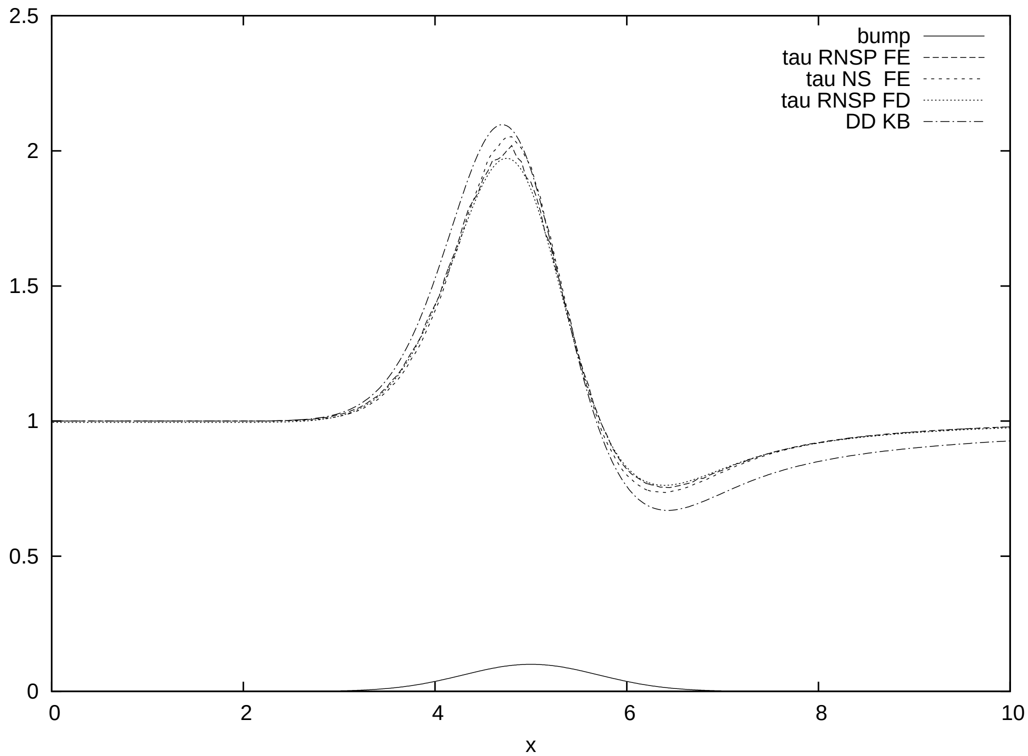


Figure 6: Skin friction on the bump of height $\alpha = 0.1$, comparison of the RNSP (FD and FEM), NS (FEM), and DD (KB) for Reynolds $Re = 750$. In Double Deck scales the height of the bump is $\alpha Re^{1/3} = 0.91$. The RSNP results are superposed for FD and FE. The Navier Stokes solution is a little bit different: the RNSP underestimates the maximum. The Double Deck over estimates the skin friction.

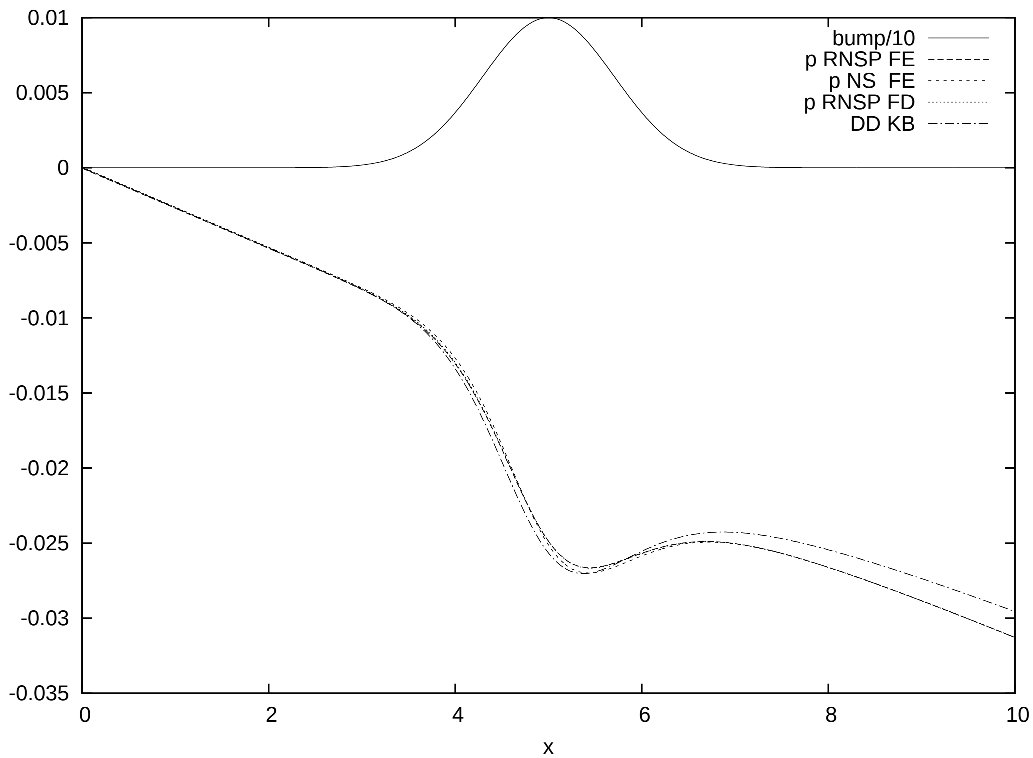


Figure 7: Pressure on the bump, comparison of the RNSP (FD and FEM), NS (FEM), and DD (KB), $Re = 750$. For the latter the basic Poiseuille pressure has been added so that $-2x/Re + Re^{-2/3}p$ is ploted. The zero value of the pressure has been put at the origin. The RNSP solutions and NS solution are superposed. The Double Deck solution differs a bit in downstream part of the bump.

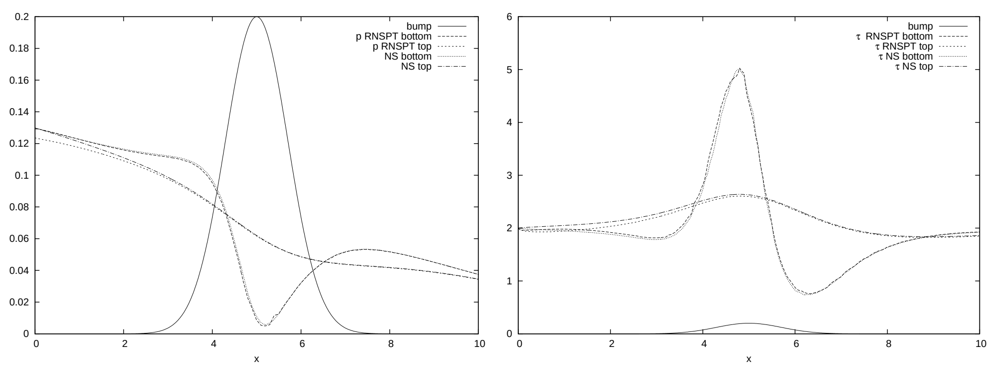


Figure 8: Comparing NS and RNSP with Transverse effects. Flat upper wall, and bumpy lower wall. Observe the upstream influence of the Skin friction and of the pressure.

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